EFFECTIVE PROPERTIES OF 1-3 PIEZOCOMPOSITES
USING EIGENSTRAIN METHOD AND MORI-TANAKA SCHEME

**Adesh K. Jain, Abdulmalik A. Alghamdi, Abhijit Dasgupta,
and James S. Sirkis**
Department of Mechanical Engineering
University of Maryland
College Park, Maryland

ABSTRACT

The focus of this paper is to obtain the effective properties of a *1-3 piezocomposite* consisting of an infinitely large isotropic host with piezoceramic (PZT) cylinders embedded along the thickness. The elastic and electric fields inside PZT cylinders are obtained using Green's function formulation. The Mori-Tanaka method is used to predict the effective properties of the 1-3 composite including elastic, piezoelectric, and dielectric constants, as functions of the PZT volume fraction.

1-INTRODUCTION

The determination of the overall properties is a classical problem in micromechanics of heterogeneous material systems. The theory of effective behavior of composite materials for uncoupled properties, for example of elastic solids (Mura, 1991), is well developed. However, effective properties for coupled phenomena, for example piezoelectric composite materials, has not been investigated in equal detail. With the emerging technology of smart structures it is important to understand the effective behavior of piezocomposites as an important material for smart structures.

Eshelby (1957) developed a unified approach to solve the elastic problem of a single ellipsoidal heterogeneity in an infinite isotropic elastic material based on imaginary cutting and welding techniques which are called "eigenstrain techniques" or transformation strain techniques. Kinoshita and Mura (1971) obtained the elastic fields for anisotropic elastic materials. Inhomogeneities made of piezoelectric materials are difficult to

handle because, I)- the elastic fields are coupled with the electric fields, and II)- the piezoelectric material behaves anisotropically for both elastic and electric fields. The coupled electric and mechanical fields for a single inclusion in a piezoelectric material have been derived for ellipsoidal heterogeneities (Wang, 1992) and for cylindrical heterogeneities (Benveniste, 1992).

The effective properties of piezoceramics have been studied by Olson and Avellaneda (1992), Wang (1992), and Benveniste (1992). Olson and Avellaneda (1992) obtained some of the effective constants of isotropic polycrystals with piezoelectric grains using effective medium approximation (EMA) and generalized Hashin-Shtrikman bounds. However, Olson and Avellaneda's (1992) method is not suitable for high concentration of voids in a material. In this paper we extend the Mori-Tanaka method for elastic solids to piezoceramics and then use the method to obtain the effective properties of piezoceramics. The single inclusion solution provided by Wang (1992) is used to solve the auxiliary problems required in the Mori-Tanaka homogenization scheme.

2-ELECTROELASTIC SINGLE INCLUSION PROBLEM

The linear coupled constitutive isothermal model for piezoelectric materials, neglecting any magnetic effects, is given by,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - h_{kij}E_k \qquad (1)$$
$$D_i = h_{ikj}\epsilon_{kj} + e_{im}E_m$$

where σ_{ij}, ϵ_{ij}, E_i, and D_i are, the stress, the strain, the electrical field, and the electrical displacement, respectively. C_{ijkl} is the elastic stiffness matrix, h_{ijk} is the piezoelectric stress coupling matrix, and e_{ij} is the dielectric permittivity matrix. Equation (1) can be written as,

$$\begin{pmatrix} \sigma \\ D \end{pmatrix} = B \begin{pmatrix} \epsilon \\ -E \end{pmatrix} \qquad (2)$$

where B is defined as,

$$B = \begin{vmatrix} C & h \\ h & -e \end{vmatrix} \qquad (3)$$

Details of matrix B are given in the appendix for a transversely isotropic material with axis 3, the piezoelectric polarization axis, as the symmetry axis. For a cylindrical inclusion in an infinite matrix, where the applied far-field electric field is E^o and strain is ϵ^o, the coupled electric and elastic fields are obtained using the Green's functions approach as (Wang, 1992)

$$\begin{pmatrix} \epsilon \\ E \end{pmatrix}^I = T \begin{pmatrix} \epsilon \\ E \end{pmatrix}^o \qquad (4)$$

where subscripts I and o denote the fields inside and outside (far-field) the inclusion, respectively. Tensor T is given in the appendix for transversely isotropic materials and cylindrical inclusions. Vector $[\epsilon\ E]^t$ is defined as,

$$\begin{pmatrix} \epsilon \\ E \end{pmatrix} = [\epsilon_{11}\ \epsilon_{22}\ \epsilon_{33}\ \epsilon_{23}\ \epsilon_{13}\ \epsilon_{12}\ E_1\ E_2\ E_3]^t \qquad (5)$$

Once the coupled elastic and electric fields are obtained, the effective elastic, dielectric, and piezoelectric constants of the piezoelectric material can be obtained using the Mori-Tanaka homogenization scheme.

3-EFFECTIVE PROPERTIES BY MORI-TANAKA METHOD

In this paper we use the equivalent inclusion average stress (EIAS) method proposed by Benveniste (1987) for elastic materials. Benveniste reformulated the Mori-Tanaka method using Eshelby's celebrated concept of equivalent inclusion. EIAS has been extended to piezoelectric ceramics (Jain, 1994), and is utilized in this paper.

The effective elastic constant $(\overline{C}_{ijkl})$, piezoelectric constants $(\overline{h}_{ijk})$, and dielectric constants $(\overline{e}_{ij})$ of the piezoelectric ceramics are defined as follows

$$<\sigma_{ij}> = \overline{C}_{ijkl} <\epsilon_{kl}> - \overline{h}_{kij} <E_k> \qquad (6)$$
$$<D_i> = \overline{h}_{ikj} <\epsilon_{kj}> + \overline{e}_{im} <E_m>$$

where the symbol $< >$ denotes the volume average of the enclosed quantity. Equation (6) can be written in terms of an effective piezo-moduli tensor $\overline{B}$ as

$$\begin{pmatrix} <\sigma> \\ <D> \end{pmatrix} = \overline{B} \begin{pmatrix} <\epsilon> \\ -<E> \end{pmatrix} \qquad (7)$$

The overall average stress, strain, electric displacement, and electric field quantities are defined as

$$<\sigma_{ij}> = \frac{1}{v}\int_v \sigma_{ij} dv = (1-v_f)\ \overline{\sigma}_{ij}^m + v_f\ \overline{\sigma}_{ij}^I = \sigma^o \qquad (8)$$

$$<\epsilon_{ij}> = (1-v_f)\ \overline{\epsilon}_{ij}^m + v_f\ \overline{\epsilon}_{ij}^I = \epsilon^o \qquad (9)$$

$$<D_i> = (1-v_f)\ \overline{D}_i^m + v_f\ \overline{D}_i^I = D^o \qquad (10)$$

$$<E_i> = (1-v_f)\ \overline{E}_i^m + v_f\ \overline{E}_i^I = E^o \qquad (11)$$

where $\overline{\sigma}_{ij}^I$, $\overline{\epsilon}_{ij}^I$, $\overline{D}_i^I$ and $\overline{E}_i^I$ are, respectively, the average stress, strain, electric displacement, and electric field in an inclusion. σ^o, ϵ^o, D^o and E^o are the constant stress, strain, electric displacement, and electric field, respectively, applied to the boundary of the domain. Superscript m denotes the respective average quantities in the matrix; v_f is the volume fraction of the heterogeneities.

The determination of the effective piezo-moduli tensor $\overline{B}$ requires expressions relating the average elastic and electric fields in the inclusions to the uniform applied elastic and electric fields on the boundary:

$$\begin{pmatrix} \epsilon \\ E \end{pmatrix}^I = A \begin{pmatrix} \epsilon \\ E \end{pmatrix}^o \qquad (12)$$

where A is termed the concentration factor and is same as tensor T in Equation (4) if it is a single inclusion problem or the interaction among inclusions is neglected (called the dilute

approximation). However, for many inclusions tensors A and **T** are not the same and the determination of the concentration factor A is a difficult task. The Mori-Tanaka approximate (Benveniste. 1987) is stated as:

$$\begin{pmatrix} \epsilon \\ E \end{pmatrix}^I = T \begin{pmatrix} \epsilon \\ E \end{pmatrix}^m \tag{13}$$

Using Equations (8) through (11),(12) and (13), we obtain concentration factor A as follows:

$$A = T \left[(1 - v_f)I + v_f T \right]^{-1} \tag{14}$$

where **I** is the unit tensor and v_f is the volume fraction of the inclusions.

Solving Equations (6) through (12), we can obtain the effective properties in terms of the concentration factor A as:

$$\bar{B} = B^m + v_f \left[(B^I - B^m)A \right] \tag{15}$$

where superscripts **m** and **I** denote quantities for matrix and inclusions, respectively. Finally, using Equation (14) the effective piezo-moduli tensor $\bar{B}$ are obtained in terms of tensor T as:

$$\bar{B} = B^m + v_f \left[(B^I - B^m)T \right] \left[(1 - v_f)I + v_f T \right]^{-1} \tag{16}$$

This approximation provides reasonable predictions of the effective properties for non-dilute concentration (Benveniste, 1987) since it simulates indirectly, the interaction among inclusions.

4-RESULTS AND DISCUSSIONS

In this section the Mori-Tanaka method is used to compute the effective properties of 1-3 piezocomposites. Figure (1) shows a schematic representation of the 1-3 piezocomposite plate consisting of piezoceramic (PZT5) rods, poled along their axis, and embedded along the thickness direction of a Stycast plate. The material properties used in this example (Avellaneda and Swart, 1994) are given in Table (I).

Figure (2) shows the change in moduli of elasticity E_1 (=E_2) (transverse) and E_3 (longitudinal) as functions of volume fraction of the piezoceramic. The longitudinal modulus varies in a linear fashion as predicted by rule-of-mixtures, while the transverse modulus shows nonlinear behavior. Figures (3) and (4) present the Poisson's ratios and shear moduli as the volume fraction of piezoceramic increases. The transverse shear modulus (G_{12}) represents the lower bound for the piezoelectric problem, analogous to Hashin's CCA model (Mura, 1991) for the elastic problem. The longitudinal shear modulus G_{23} (=G_{31}) changes sign at $V_f \approx 0.6$ because of the piezoelectric effect. The

TABLE (I): MATERIAL PROPERTIES OF PZT5 AND STYCAST.

Material	PZT5	Stycast
$S_{11}^E (10^{-12} \ m^2/N)$	16.4	108
$S_{12}^E (10^{-12} \ m^2/N)$	-5.74	-32
$S_{13}^E (10^{-12} \ m^2/N)$	-7.22	-32
$S_{33}^E (10^{-12} \ m^2/N)$	18.8	108
$d_{13} (10^{-12} \ C/N)$	-171	0
$d_{33} (10^{-12} \ C/N)$	374	0
e_{33}^T/e_o	207	4

permittivity constants are displayed in Figure (5) as functions of volume fraction of PZT5. e_{33} shows linear behavior as predicted by rule-of-mixtures, while e_{11} (=e_{22}) behaves non-linearly. Finally, the enhancements of the overall strain coupling constants (**d**) are shown in figure 6. The magnitude of d_{31} is found to attain a maximum at $V_f \approx 0.6$, unlike the predictions from the effective medium approximation (EMA) reported in the literature (Avellaneda and Swart, 1994).

6-REFERENCES

Avellaneda, M., and Swart, P. J., 1994, "Effective Moduli and Electro-acoustic Performance of Epoxy-Ceramic 1-3 Piezocomposites," *SPIE conference in Smart Structures and Materials*, H. Thomas Banks ed., Orlando, FL, SPIE Vol. 2192. pp. 394-402.

Benveniste, Y., 1987, "A New Approach to the application of Mori-Tanaka Theory in Composite Materials," *Mechanics of Materials*, Vol. 6, pp. 147-157.

Benveniste, Y., 1992, "The Determination of the Elastic and Electric Fields in a Piezoelectric Inhomogeneity," *Journal of Applied Physics*, Vol .72, No.3, pp. 1086-1095.

Eshelby, J. D., 1957, "The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems," *Proceedings of the Royal Society*, Series A, Vol. 241, pp. 376-396.

Jain, A. K., 1994, "Failure Analysis of Piezoelectric Ceramics Using Continuum Damage Mechanics," Ms. Thesis, University of Maryland, College Park, MD.

Kinoshita, N. and Mura, T., 1971, "Elastic Fields of Inclusions in Anisotropic Media, " *Physical State of Solids*, Vol. 5. pp. 759.

Mura, T. 1987, *Micromechics of Defects in Solids*. Martin Nijhoff Publications, Boston, MA.

Olson, T., and Avellaneda, M., 1992 "Effective Dielectric and Elastic Constants of Piezoelectric Polycrystal," *Journal of Applied Physics*, Vol. 71, pp. 4455-4464.

Wang, B., 1992 "Three-Dimensional Analysis of an Ellipsoidal Inclusion in a Piezoelectric Material," *International Journal of Solids and Structures*, **29**(3), pp. 293-308.

6-APPENDIX

For transversely isotropic material with axis 3 as the symmetry axis, *piezo-moduli* tensor **B** can be written in the matrix form,

$$
B = \begin{vmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & h_{31} \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & h_{31} \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & h_{33} \\
0 & 0 & 0 & C_{44} & 0 & 0 & 0 & h_{15} & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 & h_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{15} & 0 & -a_{11} & 0 & 0 \\
0 & 0 & 0 & h_{15} & 0 & 0 & 0 & -a_{11} & 0 \\
h_{31} & h_{31} & h_{33} & 0 & 0 & 0 & 0 & 0 & -a_{33}
\end{vmatrix} \tag{A1}
$$

Tensor **T** for transversely isotropic material and cylindrical inclusions can be written as,

$$
T = \begin{vmatrix}
T_{11} & T_{12} & T_{13} & 0 & 0 & 0 & 0 & 0 & T_{19} \\
T_{12} & T_{11} & T_{13} & 0 & 0 & 0 & 0 & 0 & T_{19} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & T_{44} & 0 & 0 & 0 & T_{48} & 0 \\
0 & 0 & 0 & 0 & T_{44} & 0 & T_{48} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & T_{66} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & T_{75} & 0 & T_{77} & 0 & 0 \\
0 & 0 & 0 & T_{75} & 0 & 0 & 0 & T_{77} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix} \tag{A2}
$$

details of tensor **T** are given as,

$$
T_{11} = \frac{x_1}{x_1^2 - x_2^2} \tag{A3}
$$

$$
T_{12} = - \frac{x_2}{x_1^2 - x_2^2} \tag{A4}
$$

$$
T_{13} = - \frac{C_{13}^1}{2C_{11}^o + C_{11}^1 + C_{12}^1} \tag{A5}
$$

$$
T_{19} = \frac{h_{13}^1}{2C_{11}^o + C_{11}^1 + C_{12}^1} \tag{A6}
$$

$$
T_{44} = [1 - \frac{(e_{11}^o h_{24}^1 - e_{22}^1 h_{15}^o) 2(h_{15}^o C_{44}^o - h_{24}^1 C_{44}^o)}{x_3 \, x_4}]^{-1} \frac{4(C_{44}^o e_{11}^o + (h_{15}^o)^2)}{x_3} \tag{A7}
$$

$$
T_{48} = [1 - \frac{(e_{11}^o h_{24}^1 - e_{22}^1 h_{15}^o) 2(h_{15}^o C_{44}^o - h_{24}^1 C_{44}^o)}{x_3 \, x_4}]^{-1} \frac{(e_{11}^o h_{24}^1 - e_{22}^1 h_{15}^o) 2(C_{44}^o e_{11}^o + (h_{15}^o)^2)}{x_3 \, x_4} \tag{A8}
$$

$$
T_{66} = [\, 1 + \frac{3C_{11}^o - C_{12}^o}{2C_{11}^o(C_{11}^o - C_{12}^o)} \, C_{66}^1 \,]^{-1} \tag{A9}
$$

$$
T_{75} = \frac{2 \, (h_{15}^o C_{44}^o - h_{24}^1 C_{44}^o)}{x_4} \, T_{44} \tag{A10}
$$

$$
T_{77} = \frac{2}{x_4} \, [C_{44}^o e_{11}^o + (h_{15}^o)^2 + T_{48}(h_{15}^o C_{44}^o - h_{24}^1 C_{44}^o)] \tag{A11}
$$

where,

$$
x_1 = 1 + \frac{(5C_{11}^o - 3C_{12}^o)C_{11}^1 - (C_{11}^o + C_{12}^o)C_{12}^1}{8 \, C_{11}^o \, (\, C_{11}^o - C_{12}^o)} \tag{A12}
$$

$$
x_2 = \frac{(5C_{11}^o - 3C_{12}^o)C_{12}^1 - (C_{11}^o + C_{12}^o)C_{22}^1}{8 \, C_{11}^o \, (\, C_{11}^o - C_{12}^o)} \tag{A13}
$$

$$
x_3 = 4[C_{44}^o e_{11}^o + (h_{15}^o)^2] + 2h_{15}^o h_{24}^1 + 2e_{11}^o C_{44}^1 \tag{A14}
$$

$$x_4 = 2[C^o_{44}e^o_{11}+(h^o_{15})^2] + h^o_{15}h^1_{15} + e^1_{11}C^o_{44} \qquad (A15)$$
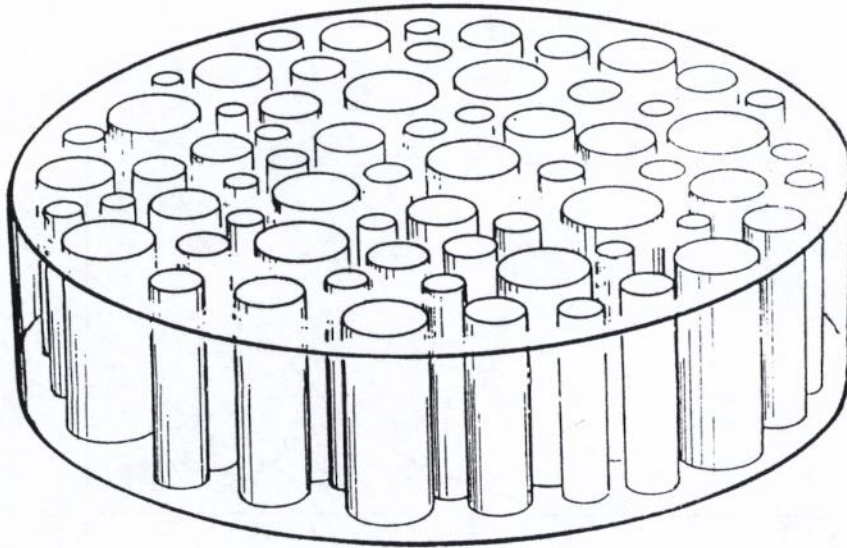
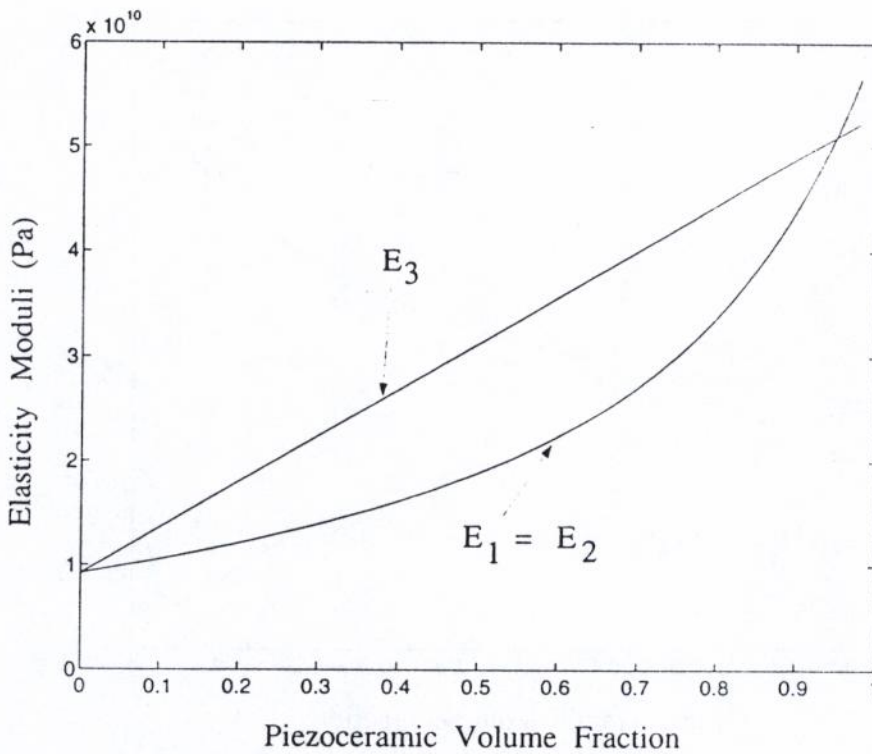FIGURE (1): SCHEMATIC REPRESENTATION OF 1-3 PIEZOCOMPOSITE.

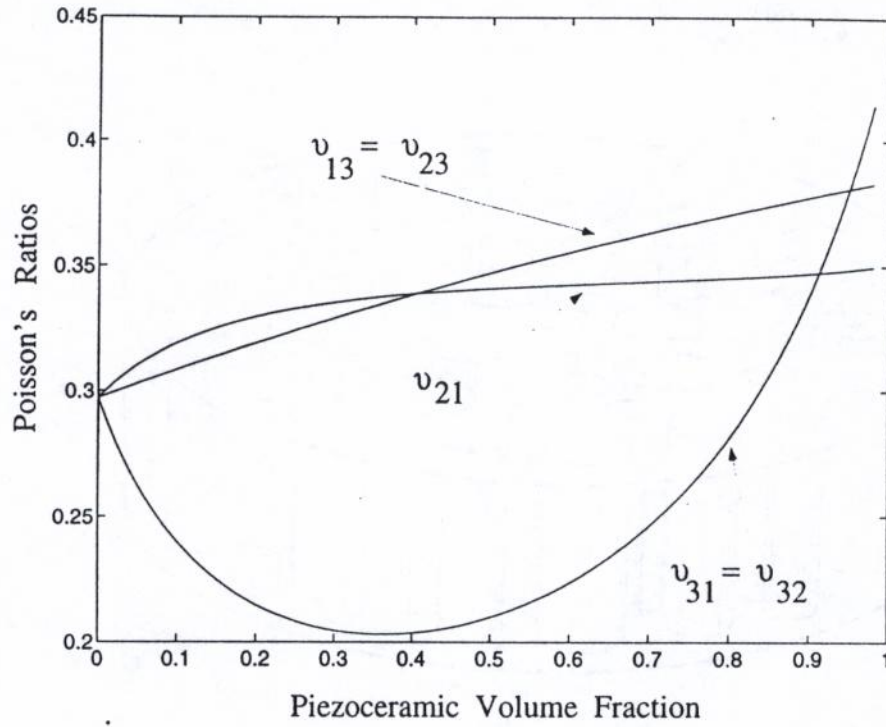FIGURE (2): MODULI OF ELASTICITY VERSUS PZT VOLUME FRACTION.

71

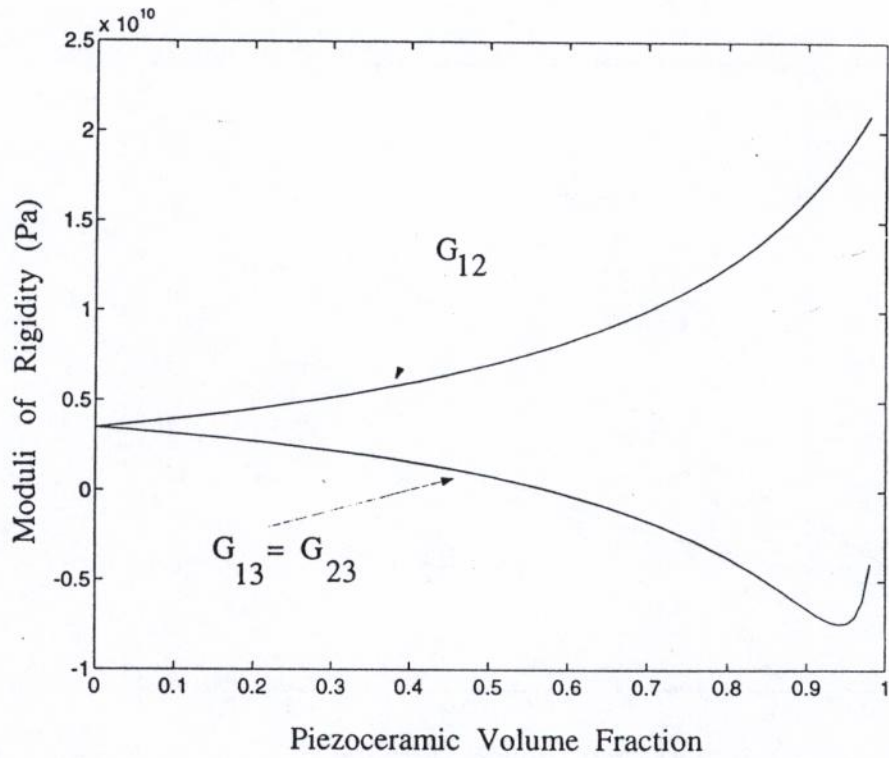FIGURE (3): POISSON'S RATIOS AS FUNCTIONS OF PZT VOLUME FRACTION.

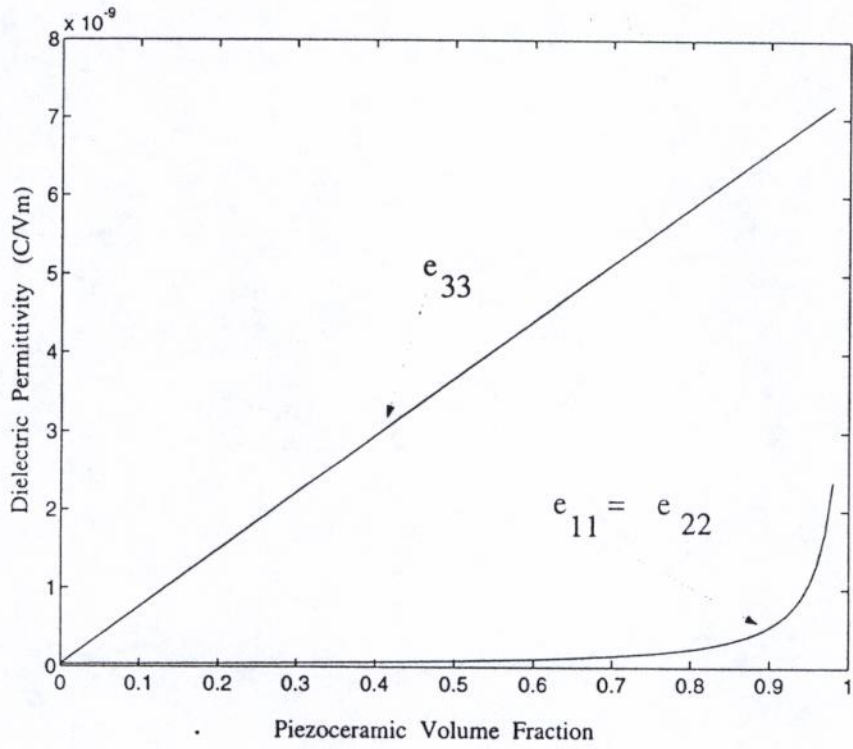FIGURE (4): SHEAR MODULI VERSUS PZT VOLUME FRACTION.

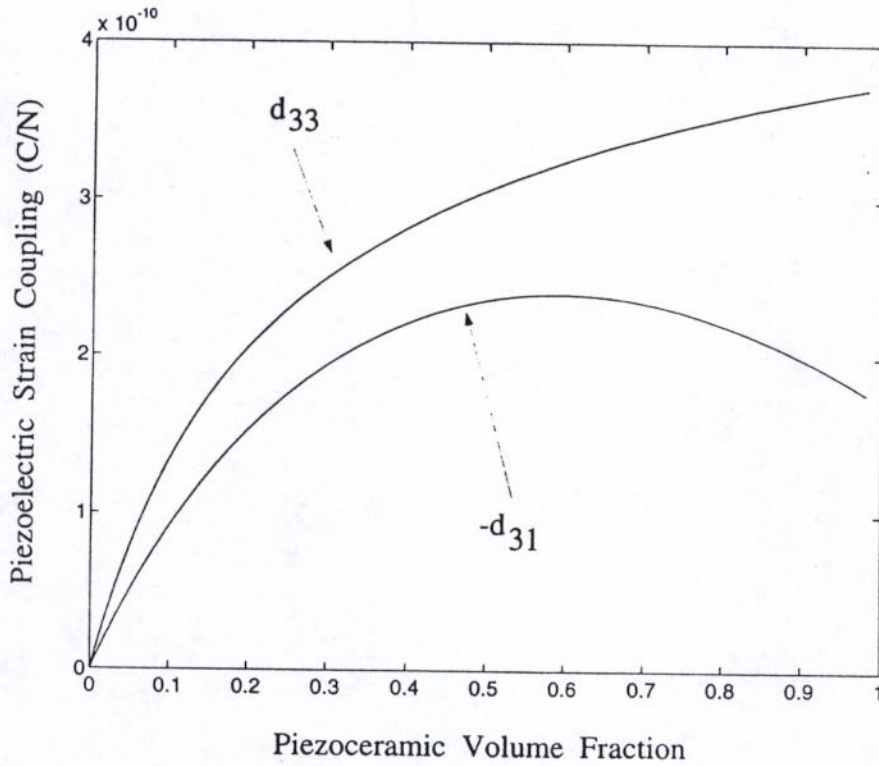FIGURE (5): THE OVERALL PERMITTIVITY CONSTANTS AS FUNCTIONS OF PZT VOLUME FRACTION.

FIGURE (6): EFFECTIVE STRAIN COEFFICIENTS VERSUS PZT VOLUME FRACTION.