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Topological modules of continuous homomorphisms

Liaqat Ali Khan

Department of Mathematics, King Abdulaziz University, PO Box 80203, Jeddah 21589, Saudi Arabia

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Abstract

In this paper, we consider the notion of module homomorphisms in the general topological module setting and establish their linearity and continuity under some suitable conditions. We also introduce the strict and uniform topologies on the modules of continuous linear homomorphisms and study their various properties.

Keywords: Topological modules over topological algebras; Module homomorphisms; Ultrabarrelled topological vector spaces; Approximate identity; Factorization theorem; Essential modules; Hypocontinuity

1. Introduction

In [21,22], Rieffel made an extensive study of the Banach module $\mathrm{Hom}_A(A, X)$ of continuous homomorphisms. Further results in this direction have been obtained by Sentilles and Taylor [25] and Ruess [24] in their study of the general strict topology. More recently, Shantha [26] has studied homomorphisms in the case of locally convex modules. The purpose of this paper is to investigate the extent to which some of the results of above authors are also true in the non-locally convex setting of topological modules.

2. Preliminaries

An algebra A (over $\mathbb{R}$ or $\mathbb{C}$) with a topology τ is called a *topological algebra* if it is a topological vector space (TVS) in which multiplication is separately continuous. A complete metrizable topological algebra is called an *F-algebra*; in this case the multiplication is jointly continuous by Arens' theorem [16, p. 24]. A net $\{e_\lambda\colon \lambda \in I\}$ in a topological algebra A is called a *left approximate identity* (respectively *right approximate identity, two-sided approximate identity*) if, for all $a \in A$, $\lim_\lambda e_\lambda a = a$ (respectively $\lim_\lambda ae_\lambda = a$, $\lim_\lambda e_\lambda a = \lim_\lambda ae_\lambda = a$); $\{e_\lambda\colon \lambda \in I\}$ is said to be *uniformly bounded* if there exists $r > 0$ such that $\{(\frac{e_\lambda}{r})^n\colon \lambda \in I,\ n = 1, 2, \dots\}$ is a bounded set in A. A TVS (E, τ) is called *ultrabarrelled* [7,9] if any linear topology τ' on E, having a base of neighbourhoods of 0 formed of τ-closed sets, is weaker than τ. (E, τ) is called *ultrabornological* [11] if every linear map from E into any TVS which takes bounded

E-mail address: akliaqat@yahoo.com.

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sets into bounded sets is continuous. Every Baire TVS (in particular, F-space) is ultrabarrelled. Every metrizable TVS is ultrabornological.

Let X be a topological vector space and A be a topological algebra, both over the same field $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$). Then X is called a *topological left A-module* if it is a left A-module and the module multiplication $(a,x)\to a.x$ from $A\times X$ into X is separately continuous. If $b(A)$ (respectively $b(X)$) denote the collection of all bounded sets in A (respectively X), then module multiplication $(a,x)\to a.x$ is called *$b(A)$-hypocontinuous* (respectively *$b(X)$-hypocontinuous*) [16, p. 28] if, given any neighbourhood G of 0 in X and any $D\in b(A)$ (respectively $B\in b(X)$), there exists a neighbourhood H of 0 in X (respectively V of 0 in A) such that $D.H\subseteq G$ (respectively $V.B\subseteq G$). Clearly, joint continuity ⇒ hypocontinuity ⇒ separate continuity; however, the converse need not hold. If E and X are TVSs, $\mathrm{BL}(E,X)$ (respectively $\mathrm{CL}(E,X)$) denotes the vector space of all bounded (respectively continuous) linear mappings from E into X. Clearly, $\mathrm{CL}(E,X)\subseteq \mathrm{BL}(E,X)$ with $\mathrm{CL}(E,X)=\mathrm{BL}(E,X)$ if E is ultrabornological (in particular metrizable). A mapping $T:E\to X$ is called a *topological isomorphism* if T is linear and a homeomorphism. If X is a left A-module, then A is said to be *faithful* in X if, for any $x\in X$, $a.x=0$ for all $a\in A$ implies that $x=0$ (cf. [12,25]).

This paper is a continuation of the author's work done in [14], but it can be read independently. In the sequel, all topological vector spaces, algebras and modules are assumed to be Hausdorff. For the general theory, the reader is referred to [7,9,30] for topological vector spaces, [16,33] for topological algebras, and [3,6,32] for topological modules.

The following result follows from [7, Theorem 7.7.3, p. 488], but we include its proof for reader's convenience and later reference.

Lemma 1. *Let (X,τ) be a topological left A-module. If X is ultrabarrelled, then the module multiplication is $b(A)$-hypocontinuous.*

Proof. Let G be a neighbourhood of 0 in X and $D\in b(A)$. For any $a\in A$, define $L_a:X\to X$ by $L_a(x)=a.x$, $a\in A$. Clearly, each L_a is linear and also continuous (by separate continuity of the module multiplication). Further, $\{L_a\colon a\in D\}$ is pointwise bounded in $\mathrm{CL}(X,X)$. [Let $x\in X$ and G_1 a neighbourhood of 0 in X. Since D is bounded in A, by separate continuity of the module multiplication, $D.x$ is bounded in X and so there exists $r>0$ such that $D.x\subseteq rG_1$. So $\{L_a(x)\colon a\in D\}=\{a.x\colon a\in D\}=D.x\subseteq rG_1$, showing that $\{L_a\colon a\in D\}$ is pointwise bounded in $\mathrm{CL}(X,X)$.] Since X is ultrabarralled, by the principle of uniform boundedness [7, p. 464], $\{L_a\colon a\in D\}$ is equicontinuous. Hence, given any neighbourhood G of 0 in X, there exists a neighbourhood H of 0 in X such that $L_a(H)\subseteq G$ for all $a\in D$; i.e. $D.H\subseteq G$ □

If (X,τ) is a topological left A-module with A having a left approximate identity $\{e_\lambda\colon \lambda\in I\}$, the *essential part X_e* of X is defined as $X_e=\{x\in X\colon e_\lambda.x\xrightarrow{\tau}x\}$ [14,21]. Clearly, $A.X\subseteq X_e$ and X_e is a topological left A-submodule of X. We say that X is *essential* if $X=X_e$.

Lemma 2. *Let (X,τ) be a topological left A-module with X ultrabarralled and A having a bounded left approximate identity $\{e_\lambda\colon \lambda\in I\}$. Then X_e is τ-closed in X.*

Proof. (Cf. [14, Theorem 8].) Let $x\in\tau\text{-cl}(X_e)$. We need to show that $e_\lambda.x\xrightarrow{\tau}x$. Let G be a neighbourhood of 0 in X. Choose a balanced neighbourhood H of 0 in X such that $H+H+H\subseteq G$. For each $\lambda\in I$, define $L_\lambda:X\to X$ by $L_\lambda(y)=L_{e_\lambda}(y)=e_\lambda.y$, $y\in X$. Since $D=\{e_\lambda\colon \lambda\in I\}$ is bounded in A, it follows from the proof of Lemma 1 that $\{L_\lambda\colon \lambda\in I\}$ is pointwise bounded and hence equicontinuous in $\mathrm{CL}(X,X)$. There exists a balanced neighbourhood H_1 of 0 in X such that

$$L_\lambda(H_1)\subseteq H \quad \text{for all } \lambda\in I.$$

Since $x\in\tau\text{-cl}(X_e)$, we can choose $x_o\in X_e$ such that

$$x-x_o\in H_1\cap H.$$

Since $e_\lambda.x_o\to x_o$ and so there exists $\lambda_o\in I$ such that

$$e_\lambda.x_o-x_o\in H \quad \text{for all } \lambda\geqslant\lambda_o.$$

Hence, for any $\lambda \geqslant \lambda_o$,

$$e_\lambda.x - x = e_\lambda.(x - x_o) + (e_\lambda.x_o - x_o) + (x_o - x) \in L_\lambda(H_1 \cap H) + H + H_1 \cap H \subseteq H + H + H \subseteq G;$$

that is, $e_\lambda.x \xrightarrow{\tau} x$ and so $x \in X_e$. Hence X_e is τ-closed. □

We now state a generalization of the famous Cohen's factorization theorem for later use. A topological algebra A is called *strongly factorable* if, for any sequence $\{a_n\}$ in A with $a_n \to 0$, there exist $b \in A$ and a sequence $\{c_n\}$ in A with $c_n \to 0$ such that $a_n = c_n b$ for all $n \geqslant 1$. A topological left A-module of X is called *A-factorable* if, for each $x \in X$, there exist $a \in A$ and $y \in X$ such that $x = a.y$. Following Ansari–Piri [2], a TVS X is called *fundamental* if there exists a constant $M > 1$ such that, for every sequence $\{x_n\}$ in X, the convergence of $M^n(x_{n+1} - x_n)$ to 0 in X implies that $\{x_n\}$ is a Cauchy sequence in X. Every locally convex and every locally bounded TVS is fundamental.

Theorem 1. *(See [2].) Let A be a fundamental F-algebra with a uniformly bounded left approximate identity. Then*:

(i) *A is strongly factorable.*
(ii) *If X is an F-space which is an essential topological left A-module, then X is A-factorable.*

We mention that if X is A-factorable, then X is essential since $X = A.X \subseteq X_e \subseteq X$, or that $X = X_e$.

Definition 1. Let E and X be topological left A-modules, where E and X are TVSs and A is a topological algebra. Then a mapping $T : E \to X$ is called an *A-module homomorphism* if $T(a.x) = a.T(x)$ for all $a \in A$ and $x \in E$ [21, p. 447]. (Similarly, if E and X are right A-modules, then we can define an A-module homomorphism as a mapping $T : E \to X$ satisfying $T(x.a) = T(x).a$ for all $a \in A$ and $x \in E$. We will state results for left modules over A, similar results holding, of course, for right modules.) At this stage, a module homomorphism is not assumed to be linear or continuous.

Our main interest is the study of A-module homomorphisms from A into X. The following algebraic result is an extension of [12, Theorem 7].

Lemma 3. *Let X be a left A-module. Suppose that A is faithful in X. Then any A-module homomorphism $T : A \to X$ is homogeneous* (*that is, $T(\lambda a) = \lambda T(a)$ for all $\lambda \in \mathbb{K}$ and $a \in A$*).

Proof. Let $a \in A$ and $\lambda \in \mathbb{K}$. Then, for any $c \in A$,

$$c.T(\lambda a) = T\big(c.(\lambda a)\big) = T\big((\lambda c)a\big) = (\lambda c).T(a) = c.\lambda T(a).$$

Since A is faithful in X, $T(\lambda a) = \lambda T(a)$. □

We now establish the linearity and continuity of an A-module homomorphisms using the factorization theorem. The following theorem extends some results in [13,15,23,27] to our more general setting.

Theorem 2. *Let X be a topological left A-module with X metrizable and A strongly factorable. Then any A-module homomorphism $T : A \to X$ is linear and continuous.*

Proof. To show that T is linear, let $a_1, a_2 \in A$ and $\alpha, \beta \in \mathbb{K}$. If we take $\{a_n\} = \{a_1, a_2, 0, 0, \ldots\}$, then clearly $a_n \to 0$; since A is strongly factorable, there exist $b, c_1, c_2 \in A$ such that $a_1 = c_1 b$, $a_2 = c_2 b$. So

$$T(\alpha a_1 + \beta a_2) = T\big((\alpha c_1 + \beta c_2)b\big) = (\alpha c_1 + \beta c_2).T(b) = \alpha T(c_1 b) + \beta T(c_2 b) = \alpha T(a_1) + \beta(a_2);$$

hence T is linear. Since X is metrizable, to show that T is continuous, it suffices to show that if $\{a_n\} \subseteq A$ with $a_n \to 0$, then $T(a_n) \to 0$. Using again the strong factorability of A, we can write $a_n = c_n b$, where $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \to 0$. Then

$$T(a_n) = T(c_n b) = c_n.T(b) \to 0.T(b) = 0$$

(by the separate continuity of module multiplication). Thus T is continuous. □

3. The topological module $\mathrm{Hom}_A(A, X)$

Definition 2. (See [21, p. 447].) Let E and X be topological left A-modules, where E and X are TVSs and A is a topological algebra. Let $\mathrm{Hom}_A(E, X)$ denote the vector space of all continuous linear left A-module homomorphisms of E into X. If E is an A-bimodule, then defining $(a * T)(x) = T(x.a)$, $\mathrm{Hom}_A(E, X)$ becomes a left A-module. In fact, for any $b \in A$, $x \in E$,

$$(a * T)(b.x) = T\big((b.x).a\big) = T\big(b.(x.a)\big) = b.T(x.a) = b.(a * T)(x).$$

In particular, $\mathrm{Hom}_A(A, X)$ is a left A-module. Note that if A is commutative, then defining $(T * a)(x) = T(a.x)$, $\mathrm{Hom}_A(E, X)$ becomes a right A-module.

We mention that $\mathrm{Hom}_A(E, X)$ has been extensively studied in the case of E and X as the Banach modules of Banach-valued function spaces $L^1(G, A)$ and $C_o(G, A)$, where G is a locally compact abelian group and A is a commutative Banach algebra (see, e.g., [8,17,20–22,28]). More recently, Abel [1] has studied it in the setting of topological bimodule-algebras. If $E = X = A$, then $\mathrm{Hom}_A(A, A)$ is the usual multiplier algebra of A, and is denoted by $M(A)$. In fact, there is a vast literature dealing with the notions of left multiplier, right multiplier, multiplier and double multiplier (see, e.g., [5,10,12,13,15,18,29]).

Lemma 4. *(Cf. [21, p. 455].) Let E and X be topological left A-modules with A having an approximate identity $\{e_\lambda\colon \lambda \in I\}$. If E is an essential A-module, then* $\mathrm{Hom}_A(E, X) = \mathrm{Hom}_A(E, X_e)$. *In particular,* $\mathrm{Hom}_A(A, X) = \mathrm{Hom}_A(A, X_e)$.

Proof. Since $X_e \subseteq X$, clearly $\mathrm{Hom}_A(E, X_e) \subseteq \mathrm{Hom}_A(E, X)$. Now let $T \in \mathrm{Hom}_A(E, X)$. Then, for any $x \in E$, since $e_\lambda.x \to x$,

$$\lim_\lambda e_\lambda.T(x) = \lim_\lambda T(e_\lambda.x) = T(x).$$

Therefore $T(x) \in X_e$, i.e. $T \in \mathrm{Hom}_A(E, X_e)$. $\square$

Definition 3. Let A be a Hausdorff topological algebra and (X, τ) be a Hausdorff TVS which is a topological left A-module and has a base $\mathcal{W}_X$ of neighbourhoods of 0 in X. The *topology of bounded convergence* $u = u_A$ (respectively the *topology of pointwise convergence* $p = p_A$) on $\mathrm{Hom}_A(A, X)$ is defined as the linear topology which has a base of neighbourhood of 0 consisting of all sets of the form

$$M(D, G) = \big\{T \in \mathrm{Hom}_A(A, X)\colon\ T(D) \subseteq G\big\},$$

where D is a bounded (respectively finite subset) of A and $G \in \mathcal{W}_X$. Clearly, $p \leqslant u$.

Further, we obtain

Lemma 5. *Let (X, τ) be a topological left A-module with $b(A)$-hypocontinuous module multiplication. Then both* $(\mathrm{Hom}_A(A, X), u)$ *and* $(\mathrm{Hom}_A(A, X), p)$ *are topological left A-modules.*

Proof. We prove the result only for $(\mathrm{Hom}_A(A, X), u)$. For any $a \in A$ and $T \in \mathrm{Hom}_A(A, X)$, the map $(a, T) \to a * T$ is separately continuous, as follows. First, let $\{a_\alpha\colon \alpha \in J\}$ be a net in A with $a_\alpha \to a \in A$, and let D be a bounded subset of A and $G \in \mathcal{W}_X$. By A-hypocontinuity, there exists a balanced $H \in \mathcal{W}_X$ such that $D.H \subseteq G$. Since T is continuous, there exists $\alpha_o \in J$ such that

$$T(a_\alpha) - T(a) \in H \quad \text{for all } \alpha \geqslant \alpha_o.$$

Now, for any $b \in D$ and $\alpha \geqslant \alpha_o$,

$$(a_\alpha * T - a * T)(b) = T(ba_\alpha) - T(ba) = b.\big[T(a_\alpha) - T(a)\big] \in D.H \subseteq G;$$

that is, $a_\alpha * T - a * T \in M(D, G)$ for all $\alpha \geqslant \alpha_o$. Hence $a_\alpha * T \xrightarrow{u} a * T$. Next, let $\{T_\alpha\colon \alpha \in J\}$ be a net in $\mathrm{Hom}_A(A, X)$ with $T_\alpha \xrightarrow{u} T \in \mathrm{Hom}_A(A, X)$, and let D be a bounded subset of A and $G \in \mathcal{W}_X$. Since the map $R_a : A \to A$ given by $R_a(b) = ba$, $b \in A$, is linear and continuous (by separate continuity of multiplication in A), it follows that $R_a(D) = Da$ is bounded in A. Since $T_\alpha \xrightarrow{u} T$, there exists $\alpha_o \in J$ such that

$$T_\alpha - T \in M(Da, G) \quad \text{for all } \alpha \geqslant \alpha_o.$$

Now, for any $b \in D$ and $\alpha \geqslant \alpha_o$,

$$(a * T_\alpha - a * T)(b) = (T_\alpha - T)(ba) \in G;$$

that is, $a * T_\alpha - a * T \in M(D, G)$ for all $\alpha \geqslant \alpha_o$. Hence $a * T_\alpha \xrightarrow{u} a * T$. □

The following results generalize some results of [5,15,19,31] to modules of continuous homomorphisms.

Theorem 3. *Let X be a topological left A-module. Then*:

(i) *If X is an F-space and A is strongly factorable, then both* $(\mathrm{Hom}_A(A, X), u)$ *and* $(\mathrm{Hom}_A(A, X), p)$ *are complete.*
(ii) *If X is complete and A is ultrabarrelled having a bounded approximate identity, then both* $(\mathrm{Hom}_A(A, X), p)$ *and* $(\mathrm{Hom}_A(A, X_e), p)$ *are complete.*

Proof. (i) Let $\{T_\alpha\colon \alpha \in J\}$ be a u-Cauchy net in $\mathrm{Hom}_A(A, X)$. Since $p \leqslant u$, $\{T_\alpha\colon \alpha \in J\}$ is a p-Cauchy net in $\mathrm{Hom}_A(A, X)$; in particular, for each $a \in A$, $\{T_\alpha(a)\}$ is a Cauchy net in A. Consequently, by completeness of X, the mapping $T : A \to X$, given by $T(a) = \lim_\alpha T_\alpha(a)$ $(a \in A)$, is well defined. Further, for any $a, b \in A$,

$$T(ab) = \lim_\alpha T_\alpha(ab) = \lim_\alpha a.T_\alpha(b) = a.T(b).$$

Since X is metrizable and A strongly factorable, by Theorem 2, $T \in \mathrm{Hom}_A(A, X)$. We now show that $T_\alpha \xrightarrow{u} T$. Let D be a bounded subset of A and take closed $G \in \mathcal{W}_X$. There exists an index α_o such that

$$T_\alpha(a) - T_\gamma(a) \in G \quad \text{for all } a \in D \text{ and } \alpha, \gamma \geqslant \alpha_o.$$

Since G is closed, fixing $\alpha \geqslant \alpha_o$ and taking $\lim_\gamma$, we have

$$T_\alpha(a) - T(a) \in G \quad \text{for all } a \in D.$$

Hence, for any $\alpha \geqslant \alpha_o$, $T_\alpha - T \in M(D, G)$. Thus $(\mathrm{Hom}_A(A, X), u)$ is complete. By a similar argument, $(\mathrm{Hom}_A(A, X), p)$ is also complete.

(ii) We first show that $\mathrm{Hom}_A(A, X)$ is p-complete. Let $\{T_\alpha\colon \alpha \in J\}$ be a p-Cauchy net in $\mathrm{Hom}_A(A, X)$. Then, for each $a \in A$, $\{T_\alpha(a)\colon \alpha \in J\}$ is a Cauchy net in X_e and hence in X for all $a \in A$. Since (X, τ) is complete, $\{T_\alpha(a)\}$ is convergent for all $a \in A$. Define $T : A \to X$ by $T(a) = \lim_\alpha T_\alpha(a)\}$, $a \in A$. Then T is a left A-module homomorphism. Also $\{T_\alpha\colon \alpha \in J\}$ is pointwise bounded. Since A is ultrabarrelled, by the principle of uniform boundedness, $\{T_\alpha\colon \alpha \in J\}$ is equicontinuous. Therefore, given any closed $G \in \mathcal{W}_X$, there exists a neighbourhood U of 0 in A such that

$$T_\alpha(U) \subseteq G \quad \text{for all } \alpha \in J; \quad \text{that is,} \quad \bigcup_{\alpha \in J} T_\alpha(U) \subseteq G.$$

If $a \in U$, $T_\alpha(a) \in T_\alpha(U) \subseteq G$. Therefore, $\{T_\alpha(a)\colon \alpha \in J\}$ is a net in G for all $a \in U$. Thus $T(a) = \lim_\alpha T_\alpha(a) \in \text{cl-}G = G$; that is, $T(U) \subseteq G$. Therefore $T : A \to X$ is continuous, and so $T \in \mathrm{Hom}_A(A, X)$. Thus $\mathrm{Hom}_A(A, X)$ is p-complete. Since A has an approximate identity, by Lemma 4, $\mathrm{Hom}_A(A, X) = \mathrm{Hom}_A(A, X_e)$. Hence $\mathrm{Hom}_A(A, X_e)$ is also p-complete. □

Definition 4. For any $x \in X$, define $R_x : A \to X$ by $R_x(a) = a.x$, $a \in A$. Clearly, R_x is linear and continuous (by separate continuity of module multiplication); further, for any $a, b \in A$,

$$R_x(ab) = (ab).x = a.(b.x) = a.R_x(b),$$

so that $R_x \in \mathrm{Hom}_A(A, X)$. Consider a map $\mu : X \to \mathrm{Hom}_A(A, X)$, given by $\mu(x) = R_x$, $x \in X$. It is easily seen that $\mu(X) = \{R_x\colon x \in X\}$ is a left A-submodule of $\mathrm{Hom}_A(A, X)$. [In fact, for any $a \in A$ and $x \in X$,

$$(a * R_x)(b) = R_x(ba) = b.(a.x) = R_{a.x}(b), \quad b \in A,$$

so that $a * R_x \in \mathrm{Hom}_A(A, X)$.]

Theorem 4. *Let X be a topological left A-module with A having a two-sided approximate identity $\{e_\lambda\colon \lambda \in I\}$. Then $\mu(X_e)$ is p-dense in $\mathrm{Hom}_A(A, X_e)$; in particular, $\mu(X)$ is p-dense in $\mathrm{Hom}_A(A, X_e)$.*

Proof. Let $T \in \mathrm{Hom}_A(A, X_e)$. For each $\lambda \in I$, define $x_\lambda = T(e_\lambda)$. Then

$$\lim_\gamma e_\gamma.x_\lambda = \lim_\gamma e_\gamma.T(e_\lambda) = \lim_\gamma T(e_\gamma e_\lambda) = T(e_\lambda) = x_\lambda,$$

and so $x_\lambda \in X_e$. Now, for any $a \in A$, $a.x_\lambda = a.T(e_\lambda) = T(ae_\lambda) \to a$; hence

$$\mu(x_\lambda)(a) = R_{x_\lambda}(a) = a.x_\lambda = a.T(e_\lambda) = T(ae_\lambda) \to T(a).$$

Therefore $\mu(x_\lambda) \xrightarrow{p} T$. Thus $T \in p\text{-cl}[\mu(X_e)]$; that is, $\mu(X_e)$ is p-dense in $\mathrm{Hom}_A(A, X_e)$. □

Remark 1. Note that $\mu(X)$ need not be u-closed in $\mathrm{Hom}_A(A, X)$ even if $X = A$ is a metrizable locally C^*-algebra (see [18, p. 187]) or a Banach algebra [31, p. 1138]. However, if $X = A$ is a B^*-algebra or, more generally, an F-algebra whose topology is generated by a submultiplicative F-norm q (cf. [33, p. 8]) such that $q(e_\lambda) = 1$ for all $\lambda \in I$, then $\mu : A \to (\mathrm{Hom}_A(A, A), u)$ is an isometry, as follows: Let $x \in X = A$. Then

$$\|R_x\|_q = \sup_{a\neq 0} \frac{q(R_x(a))}{q(a)} = \sup_{a\neq 0} \frac{q(ax)}{q(a)} \leqslant \sup_{a\neq 0} \frac{q(a)q(x)}{q(a)} = q(x).$$

On the other hand,

$$\|R_x\|_q = \sup_{a\neq 0} \frac{q(ax)}{q(a)} \geqslant \frac{q(xe_\lambda)}{q(e_\lambda)} \geqslant q(xe_\lambda) \quad \text{for all } \lambda \in I;$$

so

$$\|R_x\|_q \geqslant \lim_\lambda q(e_\lambda x) = q\Big(\lim_\lambda e_\lambda x\Big) = q(x).$$

Hence $\|R_x\|_q = q(x)$. Therefore μ is an isometry; hence A is u-closed in $\mathrm{Hom}_A(A, A)$.

To study further properties of the map $\mu : X \to \mathrm{Hom}_A(A, X)$, we need to consider the *strict topology* $\beta = \beta_A$ on X, defined originally by Sentilles and Taylor [25] and studied recently in [14,26].

Definition 5. Let (X, τ) be a topological left A-module, where A is a topological algebra, and let $\mathcal{W}_X$ be a base of τ-neighbourhoods of 0 in X. For any bounded set $D \subseteq A$ and $G \in \mathcal{W}_X$, we set

$$N(D, G) = \{x \in X\colon D.x \subseteq G\}.$$

The *uniform topology* $\tau' = \tau'_A$ (respectively *general strict topology* $\beta = \beta_A$) on X is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form $N(D, G)$, where D is a bounded (respectively finite) subset of A and $G \in \mathcal{W}_X$.

It is shown in [14] that (i) $\beta \leqslant \tau'$; (ii) if the module multiplication is $b(A)$-hypocontinuous (in particular, X is ultrabarrelled), then $\tau' \leqslant \tau$; (iii) if A has a two-sided approximate identity (and (X, τ) is Hausdorff), then β and τ' are Hausdorff. Note that, for any bounded set $D \subseteq A$ and $G \in \mathcal{W}_X$, we have

$$M(D, G) \cap \mu(X) = \big\{T \in \mathrm{Hom}_A(A, X)\colon T(D) \subseteq G\big\} \cap \mu(X) = \big\{R_x\colon x \in X,\ R_x(D) \subseteq G\big\} = \mu\big(N(D, G)\big);$$

hence τ' is the topology of bounded convergence of $\mathrm{Hom}_A(A, X_e)$ induced on X under the algebraic embedding μ.

Considering $Y = \mathrm{Hom}_A(A, X)$ as a left A-module, we can also define the *strict topology* $\beta = \beta_A$ on $\mathrm{Hom}_A(A, X)$ as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(D, G) = \{y \in Y:\ D * y \subseteq G\},$$

where D is a finite subset of A and $G \in \mathcal{W}_Y$. We mention that if $X = A = C_o(S)$ with S a locally compact Hausdorff space, then A is a commutative Banach algebra having a bounded approximate identity and $\mathrm{Hom}_A(A, A) = M(A) = C_b(S)$ [31]. In this case, β is the original strict topology on $C_b(S)$ introduced by R.C. Buck [4] in 1958.

In the remaining part of this paper, we investigate the completeness of both (X, β) and $(\mathrm{Hom}_A(A, X), \beta)$. These results extend the corresponding results of [15,24–26].

Theorem 5. *Let X be a topological left A-module with A having a two-sided approximate identity $\{e_\lambda:\ \lambda \in I\}$. If (X, β) is complete, the map $\mu : X \to \mathrm{Hom}_A(A, X_e)$ defined by $\mu(y) = R_y$, $y \in X$, is onto.*

Proof. By Theorem 4, $\mu(X)$ is p-dense in $\mathrm{Hom}_A(A, X_e)$. We now show that $\mu(X)$ is p-closed in $\mathrm{Hom}_A(A, X_e)$. Let $T \in p\text{-cl}\,\mu(X)$. There exists a net $\{x_\alpha:\ \alpha \in J\} \subseteq X$ such that $R_{x_\alpha} \xrightarrow{p} T$. Then $\{x_\alpha:\ \alpha \in J\}$ is β-Cauchy in X. [Let D be a finite subset of A and $G \in \mathcal{W}_X$. Choose a balanced $H \in \mathcal{W}_X$ with $H + H \subseteq G$. Since $R_{x_\alpha} \xrightarrow{p} T$, there exists $\alpha_o \in I$ such that for all $\alpha \geqslant \alpha_o$,

$$R_{x_\alpha} - T \in N(D, H) \quad \text{or} \quad R_{x_\alpha}(a) - T(a) \in H \quad \text{for all } a \in D.$$

Then, for any $a \in D$ and $\alpha, \gamma \geqslant \alpha_o$,

$$a.x_\alpha - a.x_\gamma = \bigl[R_{x_\alpha}(a) - T(a)\bigr] + \bigl[T(a) - R_{x_\gamma}(a)\bigr] \in H + H \subseteq G.]$$

Since (X, β) is complete, $x_\alpha \xrightarrow{\beta} x_o$, $x_o \in X$. Hence $R_{x_\alpha} \xrightarrow{p} R_{x_o}$. By uniqueness of limit in Hausdorff spaces, $T = R_{x_o} \in \mu(X)$. Thus $\mu(X) = \mathrm{Hom}_A(A, X_e)$. □

Theorem 6. *Let X be a left A-module with X complete and A ultrabarrelled having a bounded approximate identity. If the map $\mu : X \to \mathrm{Hom}_A(A, X_e)$ is onto, then (X, β) is complete.*

Proof. Let $\{x_\alpha\}$ be a β-Cauchy net in X. Therefore $\{R_{x_\alpha}\}$ is a p-Cauchy net in $\mathrm{Hom}_A(A, X_e)$. By Theorem 3, $\mathrm{Hom}_A(A, X_e)$ is p-complete, and so $R_{x_\alpha} \xrightarrow{p} T$ in $\mathrm{Hom}_A(A, X_e)$. Since μ is onto, there exists $x_o \in X$ such that $T = R_{x_o}$. Therefore $R_{x_\alpha} \xrightarrow{p} R_{x_o}$. Now, let D be a finite subset of A and $G \in \mathcal{W}_X$. Since $R_{x_\alpha} \xrightarrow{p} R_{x_o}$, there exists $\alpha_o \in I$ such that for all $\alpha \geqslant \alpha_o$,

$$R_{x_\alpha} - R_{x_o} \in M(D, G) \quad \text{or} \quad x_\alpha - x_o \in N(D, G).$$

Hence $x_\alpha \xrightarrow{\beta} x_o$, and so (X, β) is complete. □

Theorem 7. *Let (X, τ) be a topological left A-module with $b(A)$-hypocontinuous module multiplication. Suppose (X, τ) is complete and A has a bounded approximate identity $\{e_\lambda:\ \lambda \in I\}$. Then $([\mathrm{Hom}_A(A, X_e)]_e, u)$ is topologically isomorphic to (X_e, τ).*

Proof. We first show that, for any $x \in X_e$, $R_x \in [\mathrm{Hom}_A(A, X_e)]_e$, or equivalently that $e_\lambda * R_x \xrightarrow{u} R_x$. Let D be a bounded subset of A and take closed $G \in \mathcal{W}_X$. By $b(A)$-hypocontinuity, there exists a balanced $H \in \mathcal{W}_X$ such that $D.H \subseteq G$. Since $x \in X_e$, $e_\lambda.x \to x$ and so there exists $\lambda_o \in I$ such that

$$e_\lambda.x - x \in H \quad \text{for all } \lambda \geqslant \lambda_o.$$

Hence, for any $a \in D$ and $\lambda \geqslant \lambda_o$,

$$(e_\lambda * R_x - R_x)(a) = R_x(ae_\lambda - a) = (ae_\lambda - a).x = a.(e_\lambda.x - x) \in D.H \in G.$$

Thus $R_x \in [\mathrm{Hom}_A(A, X_e)]_e$.

In view of the above, we can define a map $\mu : X_e \to [\mathrm{Hom}_A(A, X_e)]_e$ by $\mu(x) = R_x$, $x \in X_e$. Clearly, μ is linear. To see that μ is one–one, let $x \in X_e$ with $\mu(x) = 0$. Since $e_\lambda.x \to x$,

$$x = \lim_\lambda e_\lambda.x = \lim_\lambda R_x(e_\lambda) = \lim_\lambda \mu(x)(e_\lambda) = 0.$$

To show that μ is onto, let $T \in [\mathrm{Hom}_A(A, X_e)]_e$. Since $e_\lambda * T \xrightarrow{u} T$, $\{e_\lambda * T\colon \lambda \in I\}$ is a u-Cauchy net in $[\mathrm{Hom}_A(A, X_e)]_e$. Now $\{T(e_\lambda)\colon \lambda \in I\}$ is a τ-Cauchy net in X. [Let $G \in \mathcal{W}_X$ be closed. If $D = \{e_\lambda\colon \lambda \in I\}$, a bounded set in A, then $M(D, G)$ is a u-neighbourhood of 0 in $[\mathrm{Hom}_A(A, X_e)]_e$; hence there exists $\lambda_o \in I$ such that

$$e_\lambda * T - e_\gamma * T \in N(D, G) \quad \text{for all } \lambda, \gamma \geqslant \lambda_o.$$

Then, for any $\alpha \in I$,

$$e_\alpha.\big[T(e_\lambda) - T(e_\gamma)\big] = T(e_\alpha e_\lambda) - T(e_\alpha e_\gamma) = (e_\lambda * T - e_\gamma * T)(e_\alpha) \in G \quad \text{for all } \lambda, \gamma \geqslant \lambda_o.$$

Therefore $T(e_\lambda) - T(e_\gamma) \in X_e$; hence $e_\alpha.[T(e_\lambda) - T(e_\gamma)] \to T(e_\lambda) - T(e_\gamma)$ and then

$$T(e_\lambda) - T(e_\gamma) \in \text{cl-}G = G \quad \text{for all } \lambda, \gamma \geqslant \lambda_o.$$

This shows that $\{T(e_\lambda)\colon \lambda \in I\}$ is a τ-Cauchy net in X_e and hence in X.] Since (X, τ) is complete, $\lim_\lambda T(e_\lambda) = z$ exists in X. For any $\gamma \in I$,

$$e_\gamma.z = e_\gamma.\lim_\lambda T(e_\lambda) = \lim_\lambda e_\gamma.T(e_\lambda) = \lim_\lambda T(e_\gamma e_\lambda) = T(e_\gamma);$$

hence $\lim_\gamma e_\gamma.z = \lim_\gamma T(e_\gamma) = z$, showing that $z \in X_e$. Now, for any $a \in A$,

$$T(a) = \lim_\lambda T(ae_\lambda) = \lim_\lambda a.T(e_\lambda) = a.z = R_z(a).$$

Consequently, $T = R_z = \mu(z)$, and so μ is onto.

To prove that μ is continuous, let $\{x_\alpha\colon \alpha \in J\}$ be a net in X_e with $x_\alpha \xrightarrow{\tau} x \in X_e$, and let D be a bounded subset of A and $G \in \mathcal{W}_X$. By $b(A)$-hypocontinuity, there exists $H \in \mathcal{W}_X$ such that $D.H \subseteq G$. Since $x_\alpha \xrightarrow{\tau} x$, there exists $\alpha_o \in J$ such that

$$x_\alpha - x \in H \quad \text{for all } \alpha \geqslant \alpha_o.$$

Then, for any $a \in D$ and $\alpha \geqslant \alpha_o$,

$$\big[\mu(x_\alpha) - \mu(x)\big](a) = [R_{x_\alpha} - R_x](a) = a.(x_\alpha - x) \in D.H \in G;$$

that is, $\mu(x_\alpha) - \mu(x) \in M(D, G)$, and so μ is continuous.

To prove that μ is open, let W be any open set in X_e and let $S \in \mu(W)$.Then there exists $y \in W$ such that $S = \mu(y) = R_y$. Since W is open, there exists a closed $G \in \mathcal{W}_X$ such that

$$y \in y + G \cap X_e \subseteq W.$$

Now, if $D = \{e_\lambda\colon \lambda \in I\}$, a bounded set in A, then $M(D, G)$ is a u-neighbourhood of 0 in $[\mathrm{Hom}_A(A, X_e)]_e$. We claim that

$$S + M(D, G) \subseteq \mu(W).$$

[Let $T \in S + M(D, G)$. Using the argument in proving μ is onto, we can write $T = R_z$, where $z = \lim_\lambda T(e_\lambda)$. Since $T - S \in M(D, G)$, for any $\lambda \in I$,

$$e_\lambda(z - y) = R_z(e_\lambda) - R_y(e_\lambda) = T(e_\lambda) - S(e_\lambda) \in G.$$

Therefore

$$z - y = \lim_\lambda e_\lambda(z - y) \in \text{cl-}G = G.$$

Hence $z \in y + G \subseteq W$. Therefore $T = R_z = \mu(z) \in \mu(W)$.] Hence $\mu(W)$ is open in $[\mathrm{Hom}_A(A, X_e)]_e$. Thus $[\mathrm{Hom}_A(A, X_e)]_e \cong X_e$. $\square$

Theorem 8. *Let (X,τ) be a topological left A-module with $b(A)$-hypocontinuous module multiplication. Suppose (X,τ) is complete ultrabarrelled and A is ultrabarrelled and ultrabornological as a TVS and has a uniformly bounded two-sided approximate identity $\{e_\lambda\colon \lambda \in I\}$. Then $(\mathrm{Hom}_A(A, X_e), \beta)$ is complete.*

Proof. Let $Y = \mathrm{Hom}_A(A, X_e)$. Since X ultrabarralled and A has a bounded left approximate identity, by Lemma 2, X_e is τ-closed and hence τ-complete. Then, by [7, p. 87], $\mathrm{BL}(A, X_e)$ is u-complete. Since A is ultrabornological, $\mathrm{BL}(A, X_e) = \mathrm{CL}(A, X_e)$ and so it follows that $Y = \mathrm{Hom}_A(A, X_e)$ is u-complete. Then, by Theorem 7, the map $\mu : X_e \to [\mathrm{Hom}_A(A, X_e)]_e = Y_e$ given by $\mu(x) = R_x, x \in X_e$, is a topological isomorphism. We observe that given $T \in Y_e$, there exists $z \in X_e$, $z = \lim_\lambda T(e_\lambda)$ and $T = T_z$.

Define a map $\sigma : Y \to \mathrm{Hom}_A(A, Y_e)$ by $\sigma(T)(a) = a * T, a \in A, T \in Y$. We need to show that σ is onto. Let $S \in \mathrm{Hom}_A(A, Y_e)$. Then $S(a) \in Y_e$ for all $a \in A$. Define $R : A \to X_e$ by

$$R(a) = \lim_\lambda S(a)(e_\lambda), \quad a \in A.$$

We claim that $R \in \mathrm{Hom}_A(A, X_e)$. [Since $S(a) \in Y_e$, $\lim_\lambda S(a)(e_\lambda)$ exists in X_e. Clearly, R is linear. Further:

$$R(ab) = \lim_\lambda S(ab)(e_\lambda) = \lim_\lambda\bigl[a.S(b)\bigr](e_\lambda) = \lim_\lambda S(b)(e_\lambda a) = S(b)(a),$$
$$a.R(b) = a.\lim_\lambda S(b)(e_\lambda) = \lim_\lambda a.S(b)(e_\lambda) = \lim_\lambda S(b)(ae_\lambda) = S(b)(a);$$

hence R is a left A-homomorphism of A into X_e. We next show that R is continuous. First, for each $\lambda \in I$, define $S_\lambda : A \to X_e$ by

$$S_\lambda(a) = S(a)(e_\lambda), \quad a \in A.$$

Clearly, each S_λ is a linear map; further, by continuity of S and separate continuity of module multiplication, S_λ is also continuous. Now $R(a) = \lim_\lambda S(a)(e_\lambda) = \lim_\lambda S_\lambda(a)$ and A is ultrabarrelled, it follows from uniform boundedness principal that R is continuous. Therefore $R \in \mathrm{Hom}_A(A, X_e)$.]

We now show that $\sigma(R) = S$. Let $a \in A$. Then, for any $b \in A$,

$$\sigma(R)(a)(b) = (a * R)(b) = R(ba) = b.\lim_\lambda S(a)(e_\lambda) = \lim_\lambda b.S(a)(e_\lambda) = \lim_\lambda S(a)(be_\lambda) = S(a)(b).$$

Thus $\sigma : Y \to \mathrm{Hom}_A(A, Y_e)$ is onto. Consequently, by Theorem 6, (Y, β) is complete. □

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