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On the relationship between Laplace transform and new integral transform "Tarig Transform"

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ABSTRACT

In this paper we discusses some relation ship between Laplace transform and the new integral transform called Tarig transform, we solve first and second order ordinary differential equations with constant and non-constant coefficients , using both transforms ,and showing Tarig transform is closely connected with Laplace transform.

Introduction

The integral transform method is an efficient method to solve, differential equations ,system of differential equations, integral equations ,system, of integral equations and so on .Recently, Tarig M. ELzaki introduced a new transform and named as Tarig transform , which is defined by the following formula.

$$E_1(u)=T[f(t)]=\frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u^2}}dt, \quad u\neq 0 \quad (1)$$

While Lapace transform is defined by the following formula

$$L[f(t)]=\int_0^\infty e^{-st}f(t)dt=F(s), \mathrm{Re}(s)>0$$

The sufficient conditions for the existence of Tarig transform are that $f(t)$ be piecewise continuous and of exponential order, this means that Tarig transform may or may not exist.

Tarig transform can certainly treat all problems that are usually treated by the well-known and extensively used Laplace transform.

Indeed as the next theorem shows Tarig transform is closely connected with Laplace transform.

Theorem (1)

Let

$$f(t)\in A=\left\{f(t)\,\middle|\,\exists \mu,k_1,k_2>0, \text{such that } |f(t)|<\mu e^{|t|/k_j}, \text{ if } t\in(-1)^j X[0,\infty)\right\}$$ With

Laplace transform $F(s)$.Then: Tarig transform $G(u)$ of $f(t)$ is given by

$$G(u)=\frac{F\left(\frac{1}{u^2}\right)}{u} \qquad (2)$$

Proof:

Let $f(t)\in A$, then for $u\neq 0$,

$$T[f(t)]=\int_0^\infty f(ut)e^{-\frac{t}{u}}dt=G(u)$$

Let $w=ut$, then we have:

$$G(u)=\int_0^\infty f(w)e^{-\frac{w}{u^2}}\frac{dw}{u}=\frac{1}{u}\int_0^\infty e^{-\frac{w}{u^2}}f(w)dw=\frac{F\left(\frac{1}{u^2}\right)}{u}$$

Also we have that $G(1)=F(1)$ so that both Tarig and Laplace transform must coincide at $u=s=1$.

Tarig Transform of Derivatives and Integrals

Being restatement of the relation(2) will serve as our working definition, since Laplace transform of $\sin t$ is $\frac{1}{1+s^2}$, then view of (2) its Tarig transform is $\frac{u^2}{1+u^4}$. This exemplifies the duality between those two transforms.

Theorem (2):

Let $F'(s)$ *and* $G'(u)$ be Laplace Tarig transform of the derivative of $f(t)$. then:

$$(i)\ G'(u)=\frac{G(u)}{u^2}-\frac{1}{u}f(0) \qquad (ii)\ G''(u)=\frac{G(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

$$(iii)\ G^{(n)}(u)=\frac{G(u)}{u^{2n}}-\sum_{j=1}^{n}u^{2(i-n)-1}f^{(i-1)}(0)$$

Where $G^{(n)}(u)$ is Tarig transform of the nth derivative $f^{(n)}(t)$ of the function $f(t)$.

Proof:

(i) Since Laplace transform of the derivatives of $f(t)$ is

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$$F'(s) = s\,F(s) - f(0)$$

Then:

$$G(u) = \frac{1}{u}F\left(\frac{1}{u^2}\right) = \frac{1}{u}\left\{\frac{1}{u^2}F\left(\frac{1}{u^2}\right) - f(0)\right\} = \frac{1}{u^3}F\left(\frac{1}{u^2}\right) - \frac{1}{u}f(0) = \frac{G(u)}{u^2} - \frac{1}{u}f(0)$$

The generalization to nth order derivative in (iii) can be proved by using mathematical induction.

Theorem (3)

Let $G'(u)$ *and* $F'(s)$ denote Tarig and Laplace transform of the definite integral of $f(t)$ $h(t) = \int_0^t f(\tau)\,d\tau$. then:

$$G'(u) = T\left[h(t)\right] = u^2 G(u)$$

Proof:

By definition of Laplace transform.

$$F'(s) = L\left[h(t)\right] = \frac{F(s)}{s}$$

Hence

$$G'(u) = \frac{1}{u}F'\left(\frac{1}{u^2}\right) = \frac{1}{u}\left[u^2 F\left(\frac{1}{u^2}\right)\right] = uF\left(\frac{1}{u^2}\right) = u^2 G(u)$$

Theorem (4):

Let $G(u)$ is Trig transform of $f(t)$ then:

$$T\left[tf(t)\right] = \frac{1}{2}\left[u^3\frac{d}{du}G(u) + u^2 G(u)\right]$$

Proof:

By definition of Tarig transform we have:

$$\frac{d}{du}G(u) = \frac{2}{u^4}\int_0^\infty tf(t)e^{-\frac{t}{u^2}}dt - \frac{1}{u^2}\int_0^\infty f(t)e^{-\frac{t}{u^2}}dt, \text{ or } \frac{1}{u}\int_0^\infty tf(t)e^{-\frac{t}{u^2}}dt = \frac{1}{2}\left[u^3\frac{d}{du}G(u) + u^2 G(u)\right]$$

$$\text{Then:}\quad T\left[tf(t)\right] = \frac{1}{2}\left[u^3\frac{d}{du}G(u) + u^2 G(u)\right]$$

Theorem (5):

Let $G(u)$ is Tarig transform of $f(t)$ then:

$$(i)\ T\left[tf'(t)\right] = \frac{u^3}{2}\frac{d}{du}\left[\frac{G(u)}{u^2} - \frac{1}{u}f(0)\right] + \frac{u^2}{2}\left[\frac{G(u)}{u^2} - \frac{1}{u}f(0)\right]$$

$$(ii)\ T\left[tf''(t)\right] = \frac{u^3}{2}\frac{d}{du}\left[\frac{G(u)}{u^4} - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)\right] + \frac{u^2}{2}\left[\frac{G(u)}{u^4} - \frac{1}{u^3}f(0) - \frac{1}{u}f'(0)\right]$$

Proof:

(i) From theorem (4), we have:

$$T\left[tf'(t)\right] = \frac{u^3}{2}\frac{d}{du}T\left(f'(t)\right) + \frac{u^2}{2}Tf'(t) = \frac{u^3}{2}\frac{d}{du}\left[\frac{G(u)}{u^2} - \frac{1}{u}f(0)\right] + \frac{u^2}{2}\left[\frac{G(u)}{u^2} - \frac{1}{u}f(0)\right]$$

The proof of (ii) is similar to the proof of (i).

Theorem (6) (Convolution)

Let $f(t)$ *and* $g(t)$ *be in* A, having Laplace transform $F(s)$ and $G(s)$, and Tarig transform $M(u)$ and $N(u)$ .Then:

$$T\left[(f * g)(t)\right] = u\,M(u)N(u)$$

Proof:

Firs recall that Laplace transforms of $(f * g)$ is given by

$$L\left[(f * g)(t)\right] = F(s)G(s)$$

Now, since, by the duality relation (2) we have, $T\left[(f * g)(t)\right] = \frac{1}{u}L\left[(f * g)(t)\right]$ and since $M(u) = \frac{F\left(\frac{1}{u^2}\right)}{u}$, $N(u) = \frac{G\left(\frac{1}{u^2}\right)}{u}$

Traig transform of $(f * g)$ is obtained as follows:

$$T\left[(f * g)(t)\right] = \frac{F\left(\frac{1}{u^2}\right).G\left(\frac{1}{u^2}\right)}{u} = u\,M(u)N(u)$$

Example (1)

Consider the first – order ordinary differential equation,

$$\frac{dx}{dt} + Px = f(t), t > 0$$

$$x(0) = a \qquad (4)$$

Where p and a are constants and $f(t)$ is an external input function so that its Laplace and Tarig Transforms are exist.

First Solution by Laplace Transform:

$$sX(s) - x(0) + PX(s) = F(s) \Rightarrow X(s) = \frac{a}{s+p} + \frac{F(s)}{s+p}$$

Where that $X(s)$ and $F(s)$ are Laplace transform of $x(t)$ and $f(t)$. Then $x(t) = a\,e^{-pt} + L^{-1}\left[\frac{F(s)}{s+p}\right]$ Or

$$x(t) = a\,e^{-pt} + \int_0^t f(t-\tau)e^{-p\tau}d\tau$$

In particular if $f(t) = c \equiv$ constant, then the Solution of (3) becomes:

$$x(t) = \frac{c}{p} + \left(a - \frac{c}{p}\right)e^{-pt}$$

Second Solution By Tarig Transform:

Using Tarig transform of equation (3) we get

$$\frac{X(u)}{u^2} - \frac{1}{u}x(0) + PX(u) = F(u)$$

Where $X(u)$ and $F(u)$ are Tarig transform of $x(t)$ and $f(t)$, then:

$$X(u) = \frac{u^2F(u)}{1+u^2p} + \frac{au}{1+u^2p}$$

The inverse Tarig transform leads to the solution in the form.

$$x(t) = \frac{c}{p} + \left(a - \frac{c}{p}\right)e^{-pt}, \text{ When } f(t) = c$$

Example (2)

Consider the ordinary differential equation with variable coefficients (Bessl's equation).

$$ty'' + y' + ty = 0 \ , y(0) = 1 \qquad (5)$$

Solution by Laplace Transform:

$$L\left[t\,y''\right] + L\left[y'\right] + L\left[t\,y\right] = 0, \text{ and}$$

$$-\frac{d}{ds}\left[sY - sy(0) - y'(0)\right] + sY - y(0) - \frac{dY}{ds} = 0$$

$$\left(s^2+1\right)\frac{dY}{ds}+sY=0\ ,\ \left(y'(0)=c\right)\Rightarrow\frac{dY}{y}=\frac{-s}{s^2+1}ds\Rightarrow\ln Y=-\ln\sqrt{1+s^2}+\ln A$$

$Y=\dfrac{A}{\sqrt{1+s^2}}$, Where Y is Laplace transform of y inverting we find: $y(t)=AJ_0(t)$

Solution by Tarig Transform:

Take Tarig transform of equation (5) we have,

$$\frac{1}{2}u^3\frac{d}{du}\left[\frac{G(u)}{u^4}-\frac{1}{u^3}y(0)-\frac{1}{u}y'(0)\right]+\frac{1}{2}u^2\left[\frac{G(u)}{u^4}-\frac{1}{u^3}y(0)-\frac{1}{u}y'(0)\right]$$
$$+\frac{G(u)}{u^2}-\frac{1}{u}y(0)+\frac{1}{2}\left[u^3\frac{d}{du}G(u)+u^2G(u)\right]=0$$

Where $G(u)$ is Tarig transform of y. Let $y'(0)=c$, we have:

$$\left(u+u^5\right)G'(u)=\left(1-u^4\right)G(u)\quad \text{And}$$
$$\frac{G'(u)}{G(u)}=\frac{1-u^4}{u+u^5}=\frac{1}{u}-\frac{2u^3}{1+u^4}$$

Integrating two sides we get: $\ln G(u)=\ln\dfrac{Au}{\sqrt{1+u^4}}$, where A is a constant.

Inversion gives the formal solution: $y(t)=AJ_0(t)$

This is the same solution.

Example (3):

Consider the following linear integral differential equation.

$$f'(t)=\delta(t)+\int_0^t f(\tau)\cos(t-\tau)d\tau\ ,\ f(0)=1 \qquad (6)$$

Solution By Laplace Transform:

By taking Laplace transform of (6), we get.

$$sF(s)-f(0)=1+\frac{s}{s^2+1}F(s)\quad \text{Or}\quad F(s)=\frac{2}{s}+\frac{2}{s^3}$$

Apply the inverse Laplace transform to find the solution of (6) in the form:

$$f(t)=2+t^2$$

Solution by Tarig Transform:

By using Tarig Transform to eq(6)we get:

$$\frac{G(u)}{u^2}-\frac{1}{u}=\frac{1}{u}+uG(u)\left[\frac{u}{1+u^4}\right]$$

And $G(u)-2u=\dfrac{u^4}{1+u^4}G(u)$ Or $G(u)=2u+2u^5$

Inverting this equation we obtain the solution in the form: $f(t)=2+t^2$

This is the same solution.

Conclusions:

Tarig transform is a convenient tool for solving differential equations in the time domain without the need for performing an inverse Tarig transform and the connection of Tarig transform with Laplace transform goes much deeper.

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Appendix Tarig Transform of Some Functions

S.N0.	$f(t)$	$G(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$
10	$H(t-a)$	$ue^{\frac{-a}{u^2}}$
11	$\delta(t-a)$	$\frac{1}{u}e^{\frac{-a}{u^2}}$
12	te^{at}	$\frac{u^3}{(1-au^2)^2}$
13	$e^{at}\sin bt$	$\frac{bu^3}{(1-au^2)^2+b^2u^4}$
14	$\int_0^t f(\omega)d\omega$	$u^2G(u)$
15	$(f*g)(t)$	$uM(u)N(u)$
16	$e^{at}\cos bt$	$\frac{u(1-au^2)}{(1-au^2)^2+b^2u^4}$
17	$e^{at}\cosh bt$	$\frac{u(1-au^2)}{(1-au^2)-b^2u^4}$
18	$e^{at}\sin bt$	$\frac{bu^3}{(1-au^2)^2-b^2u^4}$
19	$t\sin at$	$\frac{2au^5}{(1+a^2u^4)^2}$
20	$t\cos at$	$\frac{u^3(1-a^2u^4)}{(1+a^2u^4)^2}$
21	$t\sinh at$	$\frac{2au^5}{(1-a^2u^4)^2}$
22	$t\cosh at$	$\frac{u^3(1+a^2u^4)}{(1-a^2u^4)^2}$