Explicate Series Solution for Prey-Predator Problem

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In this paper, an analytical expression for the solution of the ratio-dependent predator-prey system with constant effort harvesting by an adaptation of the homotopy perturbation method (HPM) is presented. The HPM is treated as an algorithm for approximating the solution of the problem in a sequence of time intervals, i.e HPM is converted into a hybrid numeric–analytic method. Residual error for the solution is presented.

1 Introduction

Most modelling of biological problems are characterized by systems of ordinary differential equations (ODEs). The prey is subjected to constant effort harvesting with r, a parameter that measures the effort being spent by a harvesting agency. The harvesting activity does not affect the predator population directly. It is obvious that the harvesting activity does reduce the predator population indirectly by reducing the availability of the prey to the predator. Adopting a simple logistic growth for prey population with $e > 0, b > 0$, and $c > 0$ standing for the predator death rate, capturing rate, and conversion rate, respectively, we formulate the problem as [1]

$$\frac{dx(t)}{dt} = x(t)(1 - x(t)) - \frac{bx(t)y(t)}{y(t) + x(t)} - rx(t), \quad x(t_0) = c_1, \tag{1}$$

$$\frac{dy(t)}{dt} = \frac{cx(t)y(t)}{y(t) + x(t)} - ey(t), \quad y(t_0) = c_2, \tag{2}$$

where $x(t)$ and $y(t)$ represent the fractions of population densities for prey and predator at time t, respectively. Equations (1–2) are to be solved according to biologically meaningful initial conditions $x(t) \geq 0$ and $y(t) \geq 0$.

Authors in [3] and [4] used the Adomian decomposition method (ADM) to handle the systems of prey-predator problem. Yusufoğlu and Erbaş [5] and Rafei et al. [6] employed the variational iteration method (VIM) to compute an approximation to the solution of the system of nonlinear differential equations governing the problem. Biazar [7] used the power series method (PSM) to handle the systems.

In recent years, a great deal of attention has been devoted to study HPM, which was first invented by Prof Ji-Huan He [8] for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations. HPM has successfully been applied to many situations. Chowdhury et al. present new modification of HPM by dividing the solution interval to finite number of subintervals [4].

In this paper, we are interested to find the approximate analytic solution of the system of coupled nonlinear ODEs (1) and (2) by treated the HPM as an algorithm for approximating the solution of the problem in a sequence of time intervals. Residual error for the present solution is introduced.

2 Solution procedure

Firstly, consider (1) and (2) subject to

$$x(t^*) = c_1, \quad y(t^*) = c_2. \tag{3}$$

We note that when $t^* = 0$ we have the initial condition of Eq. (1) and (2). It is straightforward to choose

$$x_0(t) = c_1, \quad y_0(t) = c_2, \tag{4}$$

as our initial approximations of $x(t)$ and $y(t)$, and the linear operator should be

$$L[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t}, \tag{5}$$

with the property

$$L[A] = 0, \tag{6}$$

where A is the integration constant, which will be determined by the initial condition.

If $q \in [0, 1]$ indicate the embedding parameter, then the *zeroth-order deformation* problems are of the following form:

$$(1-q)L[\hat{x}(t;q) - x_0(t)] = qN_x[\hat{x}(t;q), \hat{y}(t;q)], \tag{7}$$

$$(1-q)L[\hat{y}(t;q) - y_0(t)] = qN_y[\hat{x}(t;q), \hat{y}(t;q)], \tag{8}$$

subject to the initial conditions

$$\hat{x}(t^*;q) = c_1, \quad \hat{y}(t^*;q) = c_2, \tag{9}$$

in which we define the nonlinear operators N_x and N_y as

$$N_x[\hat{x}(t;q), \hat{y}(t;q)] = \frac{\partial \hat{x}(t;q)}{\partial t} - \hat{x}(t;q)(1-\hat{x}(t;q)) + \frac{b\hat{x}(t;q)\hat{y}(t;q)}{\hat{y}(t;q)+\hat{x}(t;q)} + r\hat{x}(t;q),$$

$$N_y[\hat{x}(t;q), \hat{y}(t;q)] = \frac{\partial \hat{y}(t;q)}{\partial t} - \frac{c\hat{x}(t;q)\hat{y}(t;q)}{\hat{y}(t;q)+\hat{x}(t;q)} + e\hat{y}(t;q).$$

For $q = 0$ and $q = 1$, the above *zeroth-order deformation* equations (7) and (8) have the solutions

$$\hat{x}(t;0) = x_0(t), \quad \hat{y}(t;0) = y_0(t), \tag{10}$$

and

$$\hat{x}(t;1) = x(t), \quad \hat{y}(t;1) = y(t). \tag{11}$$

When q increases from 0 to 1, then $\hat{x}(t;q)$ and $\hat{y}(t;q)$ vary from $x_0(t)$ and $y_0(t)$ to $x(t)$ and $y(t)$. Expanding $\hat{x}$ and $\hat{y}$ in Taylor series with respect to q, we have

$$\hat{x}(t;q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) q^m, \tag{12}$$

$$\hat{y}(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{13}$$

in which

$$x_m(t) = \frac{1}{m!} \frac{\partial^m \hat{x}(t;q)}{\partial q^m}\bigg|_{q=0}, \quad y_m(t) = \frac{1}{m!} \frac{\partial^m \hat{y}(t;q)}{\partial q^m}\bigg|_{q=0}. \tag{14}$$

Therefore, we have through Eq. (11) that

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t), \tag{15}$$

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{16}$$

Define the vectors

$$\vec{x}(t) = \{x_0(t), x_1(t), ..., x_n(t)\}, \tag{17}$$

$$\vec{y}(t) = \{y_0(t), y_1(t), ..., y_n(t)\}. \tag{18}$$

Differentiating the *zeroth-order* equations (7) and (8) m times with respect to q, then setting $q = 0$, and finally dividing by $m!$, we have the *mth-order deformation* equations

$$L[x_m(t) - \chi_m x_{m-1}(t)] = R_{x,m}(\vec{x}(t), \vec{y}(t)), \tag{19}$$

$$L[y_m(t) - \chi_m y_{m-1}(t)] = R_{y,m}(\vec{x}(t), \vec{y}(t)), \tag{20}$$

with the following boundary conditions:

$$x_m(t^*) = 0, \quad y_m(t^*) = 0, \tag{21}$$

for all $m \geq 1$, where

$$R_{x,m}(\vec{x}(t), \vec{y}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_x[\hat{x}(t;q), \hat{y}(t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{22}$$

$$R_{y,m}(\vec{x}(t), \vec{y}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_y[\hat{x}(t;q), \hat{y}(t;q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{23}$$

This way, it is easy to solve the linear non-homogeneous Eqs. (19) and (20) at general initial conditions by using Maple, one after the other in the order $m = 1, 2, 3, \ldots$. Thus we successfully have

$$\begin{aligned}
x_1(t) &= -\frac{c_1\,\left(-c_1 + 7\,c_2 + 10\,{c_1}^2 + 10\,c_1\,c_2\right)(t - t^*)}{10(c_1 + c_2)}, \\
y_1(t) &= -\frac{c_2\,(3\,c_1 + 5\,c_2)\,(t - t^*)}{10(c_1 + c_2)}, \\
x_2(t) &= \frac{1}{200\,(c_1 + c_2)^3}\,c_1\,\left({c_1}^3 - 30\,{c_1}^4 + 200\,{c_1}^5 + 19\,{c_1}^2 c_2\right. \\
& +19\,c_1\,{c_2}^2 + 49\,{c_2}^3 + 70\,{c_1}^3 c_2 + 310\,{c_1}^2{c_2}^2 + 210\,c_1\,{c_2}^3 \\
& \left.+600\,{c_1}^4 c_2 + 600\,{c_1}^3{c_2}^2 + 200\,{c_1}^2{c_2}^3\right)(t - t^*)^2 \\
y_2(t) &= -\frac{1}{200\,(c_1 + c_2)^3}\,c_2\,\left(-25\,{c_2}^3 - 9\,{c_1}^3 - 47\,{c_1}^2 c_2\right. \\
& \left.-51\,c_1\,{c_2}^2 + 20\,{c_1}^3 c_2 + 20\,{c_1}^2{c_2}^2\right)(t - t^*)^2, \\
& \vdots
\end{aligned}$$

By the same way we can get the first fourth term to be as analytical approximate solution as $x(t) \simeq \sum_{i=0}^{4} x_i(t)$ and $y(t) \simeq \sum_{i=0}^{4} y_i(t)$ terms. Now we divide the interval $[0, T]$ to subintervals by time step $\Delta t = 0.01$. Then we start from the initial conditions and we get the solution on the interval $[0, 0.01)$. Further, we take $c_1 = x(0.01)$ and $c_2 = y(0.01)$ and $t^* = 0.01$, so we get the solution on the new interval $[0.01, 0.02)$, and so on. Therefore, by choosing this initial approximation on the starting of each interval, the solution on the whole interval should be continuous. It is worth mentioning that if we take $t^* = 0$ and we fixed c_1 and c_2, then the solution will be the standard HPM solution which is not effective at large value of t.

3 Analysis of results

In this section, we compute the result using above algorithm for $x(0) = 0.5, y(0) = 0.3, b = 0.8, c = 0.2, e = 0.5$ and $r = 0.9$. Figure 1 presents the population fraction versus time for prey population fraction $(x(t))$ and predator population fraction $(y(t))$. Moreover the residual error using this algorithm is given in Fig.2 (a)and (b). It is clear that the error within the range 10^{-15} which mean that is very small and it is not be possible in the standard HPM which give error 10^6 using the same interval as in Fig. 2(c).

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(a) (b)

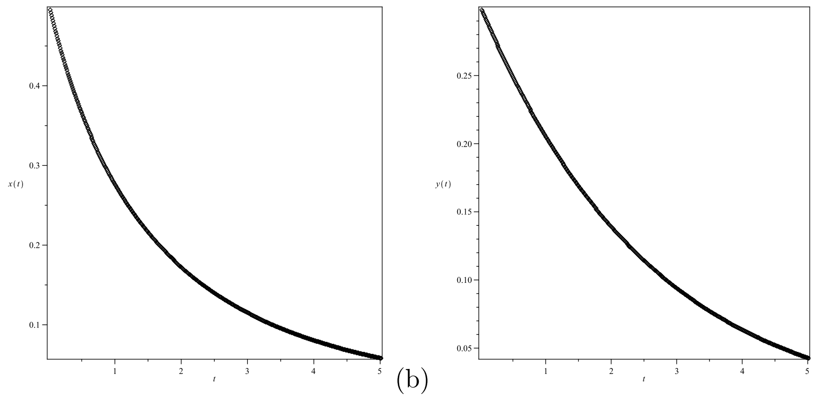

Figure 1: Population fraction versus time(a) prey population fraction; (b) predator population fraction.

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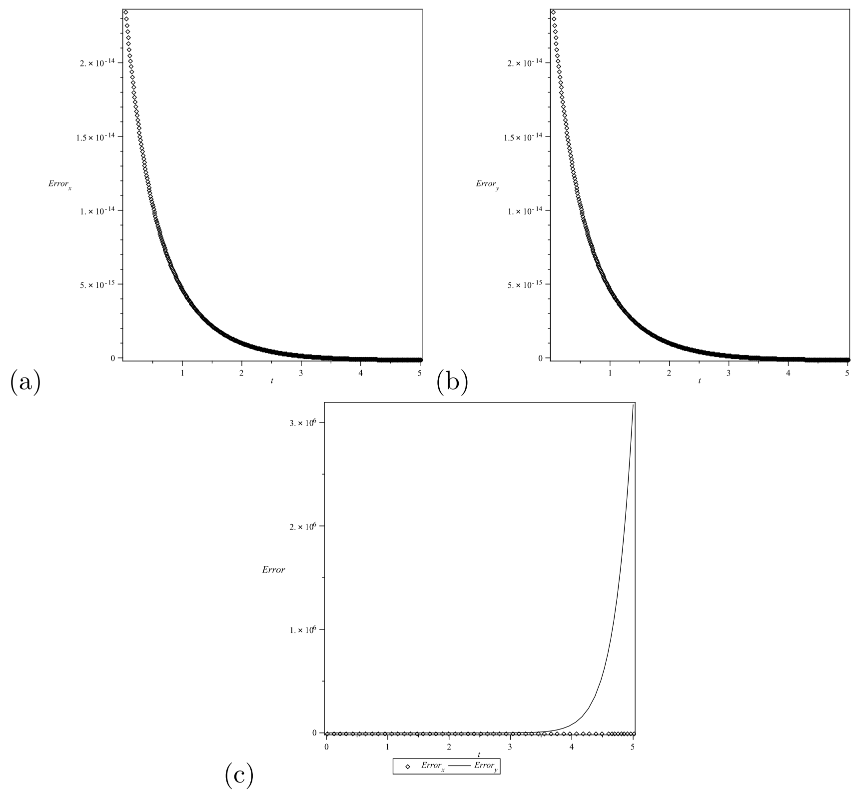

Figure 2: Residual error for (a) MHPM solution of x (b) MHPM solution of y (c) HPM solution